Join and Powers of Some Wedge Graphs as Segment Intersection Graph[1]

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ABSTRACT

In graph theory, we can always make a graph representation of a collection of objects. This is commonly done with "intersection graphs", in which every vertex is defined as a collection of objects and the two vertices are adjacent if their collections intersect. A segment intersection graph (SEG) is an intersection graph of a non-empty family $\mathscr{L}$ of line segments in the plane denoted by $\Omega(\mathscr{L})$ whose vertex-set is $\mathscr{L}$ where there is an edge between two vertices ℓ_1 and ℓ_2 in $\mathscr{L}$ if $\ell_1 \cap \ell_2 \neq \emptyset$. If $\mathscr{L}$ is a family of half-lines, $\Omega(\mathscr{L})$ is called a half-line intersection graph. It is known that recognition of such graphs is NPhard. Here, we consider an intersection graph of half-lines contained in an arbitrarily thin θ-slice of the plane (the convex subset of R^2 bounded by two half-lines with a common end-point and making an angle of θ (radians) with each other, $0 < \theta < \pi$ called wedge graphs (WEDG). We show that wedge graphs are segment intersection graphs and unit segment intersection graphs. In particular, we prove that the join and powers of some paths, cycles, fans and wheels are wedge graphs and consequently, segment intersection graphs.

Key words: *Graph theory, segment intersection graph, intersection graph*

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INTRODUCTION

All graphs considered here are finite, loopless and without multiple edges. We call the graph with only one vertex as the trivial graph. If G is a graph, we denote by V (G), the set of vertices of G; and E (G); the set of edges of G.

Given a set $\mathscr{L}$ of line segments in the plane R^z, its intersection graph denoted by Ω ($\mathscr{L}$), is the graph with a vertex for every segment and two vertices are adjacent if the corresponding line segment intersect. We call Ω ($\mathscr{L}$) a segment of intersection graph. The set of $\mathscr{L}$ of line segments is said to be the line segment representation of the graph Ω ($\mathscr{L}$). An example is the graph G shown in Figure 1. The vertices of G are represented by straight line segments and the edges of G are represented by intersection points.

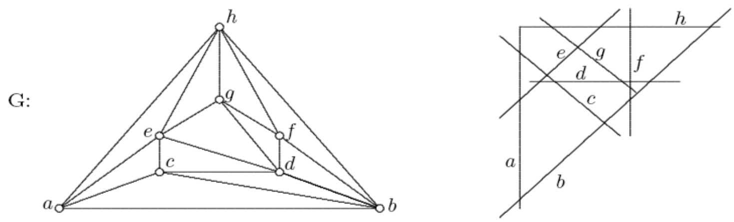


Figure 1. G as Intersection Graph of Line Segments

Note that not all graphs are segment intersection graphs. An example was given by J. Pach, et. al [11], wherein the graph was obtained from the complete graph K_5 by inserting one vertex in each of the ten edges. The general case was given by Gervacio [4], where the graph is obtained from non-planar graph N by inserting a vertex in every edge of N.

The question whether every planar graph allows a representation of straight line segments has become a well-known problem in the area of geometric graph theory. This arose from Scheinerman's conjecture [12] that all planar graphs are segment intersection graphs. The conjecture was formulated by E.R. Scheinerman in his PhD thesis (1984), following the results due to Ehrlich et.al in 1976 that every planar graph could be represented as the intersection graph of a set of simple curves in the plane. Some special cases of this conjecture have been resolved. In 1991, Hartman et.al [7] and de Fraysseix et.al [2] proved that every bipartite planar graph can be represented as the intersection graph of horizontal and vertical line segments. H. De Fraysseix and J. Kratochvil [3] writes, "The result stating that planar triangle-free graphs are intersection graphs of segments in three directions in the plane proved in Triangle-free Planar Graphs and Segment

Intersection Graphs by de Castro, Cobos, Dana, Marquez and Noy is still today, 2 years after the GD'99 conference, the strongest result in the area."

Other studies on segment intersection graphs were made by Gervacio, particularly the intersection graph of halfplanes and of line segments in the plane. In [5], he proved the characterization theorems for intersection graph of halfplanes and showed that these graphs are isomorphic to some halfline intersection graph. He also proved that every intersection graph of a finite family of halfplanes is isomorphic to some segment intersection graph. Gervacio [4] also showed that half-line intersection graphs are segment intersection graphs and proved that a finite join of trees is a half-line intersection graph.

WEDGE GRAPHS

Gervacio [5] defined a half-line as an unbounded line segment with one endpoint. An example of which is the positive x-axis.

Definition 2.1

Let $\mathscr{L}$ be a non-empty finite family of subsets of the plane R^2. The intersection graph of $\mathscr{L}$, denoted by $\Omega(\mathscr{L})$, is the graph with vertex set $\mathscr{L}$, where $\ell_1\ell_2$ is an edge if and only if $\ell_1 \neq \ell_2$ and $\ell_1 \cap \ell_2 \neq \emptyset$. If $\mathscr{L}$ is a finite family of half-lines, any graph isomorphic to $\Omega(\mathscr{L})$ is called a half-line intersection graph. In other words, G is a half-line intersection graph if there exists a family of half-lines $\mathscr{L}$, such that G is isomorphic to $\Omega(\mathscr{L})$. The family of halflines $\mathscr{L}$ is called a half-line representation of G.

The trivial graph is a half-line intersection graph. The complete graph K_n is a half-line intersection graph which can be represented by concurrent half-lines. Gervacio [4] proved the existence of a special representation of trees as a half-line intersection graph and used the result to prove that the join of a finite number of trees is a half-line intersection graph.

Definition 2.2

Let θ be any real number satisfying $0 < \theta < \pi$. By a θ-slice of the plane (or simply slice of the plane) we mean the convex subset of R^2 bounded by two distinct half-lines with a common end-point and making an angle θ (radians) with each other. We say that a slice of the plane is arbitrarily small if θ is arbitrarily small.

Definition 2.3

A graph G is a wedge graph denoted by WEDG, if and only if for every θ-slice S_θ of the plane, there exists a family $\mathscr{L}$ of half-lines satisfying the following conditions:

a.) Every $\ell \in \mathscr{L}$ is contained in S_θ.
b.) The half-lines in L together with the boundaries of S_θ are non-parallel, and
c.) $\Omega(\mathscr{L}) \cong G$.

Example 2.1

The graph shown in Figure 2 together with its half-line representation is a wedge graph for every $0 < \theta < \pi$.

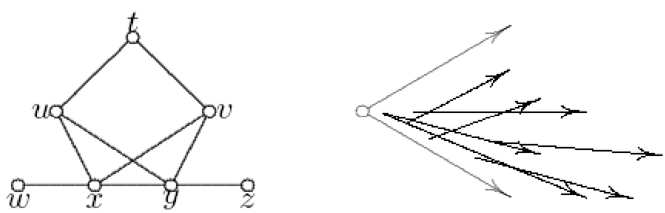


Figure 2. An example of a wedge graph

Theorem 2.2

The special graphs such as the trees, paths, stars, complete graphs and null graphs are wedge graphs.

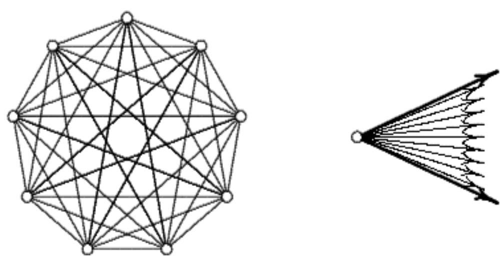


Figure 3. The Complete Graph K_9 and its Half-line Representation

Definition 2.4

The segments $\ell_1, \ell_2, \ldots, \ell_k$ ($k \geq 3$) form a closed chain if there is a k-gon whose sides are segments of $\ell_1, \ell_2 \ldots.. \ell_k$.

Example 2.2

Consider the segments $\ell_1, \ell_2, \ell_3, \ell_4$ in Figure 4 which form a closed chain of four segments.

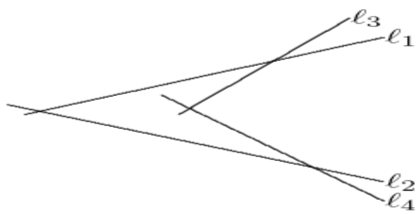


Figure 4. Segments that form a closed chain

Lemma 2.3

If $G \cong \Omega(\mathscr{S})$, and G has a cycle C_k, $k \geq 3$ as a subgraph, then there exists segments $\mathscr{S}'$ such that $G \cong \Omega(\mathscr{S}')$ and the segments $\mathscr{S}'$ representing the vertices of C_k space form a closed chain of k segments.

Theorem 2.4

The cycle and the crown are wedge graphs.

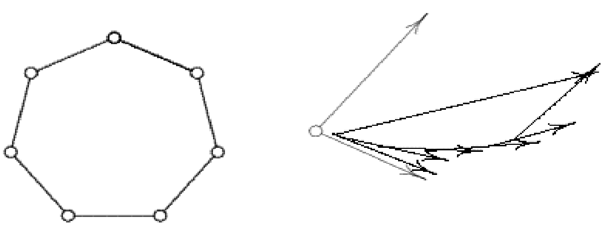


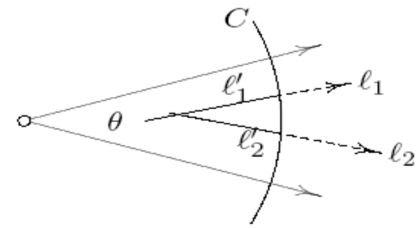
Figure 5: The cycle C_7 and its Half-line Representation

WEDGE GRAPHS AS SEGMENT INTERSECTION GRAPHS

Theorem 3.1

Every wedge graph is a segment intersection graph.

Proof. Let $\mathscr{L}$ be a non-empty, non-parallel finite family of half-lines contained in a θ-slice of the plane. Let I be the set of points of intersection of the half-lines in $\mathscr{L}$ together with the endpoints of the half-lines in $\mathscr{L}$. Since I is finite, there exists a circle C which encloses I. For each $\ell \subset \mathscr{L}$, let $i_\ell = C \cap \ell$. Let ℓ' be the bounded line segment whose end points are i_ℓ and the end-point of ℓ. Let $\mathscr{L}$ be the family of all ℓ', where ℓ ranges all the half-lines in $\mathscr{L}$. Then clearly, $\Omega(\mathscr{L}) \cong \Omega(\mathscr{L}')$. □



The converse of the Theorem is not true as in the case of the θ-graph G(5,5,5). We show here that there exist segment intersection graphs which cannot be classified as wedge graphs, thus establishing the relationship WEDG⊂SEG.

Remark 3.2

A graph G is not a half-line intersection graph if for every segment representation of G, there is a segment interior to a closed chain of line segments.

Theorem 3.3

The graph G(5,5,5) is a segment intersection graph but not a wedge graph.

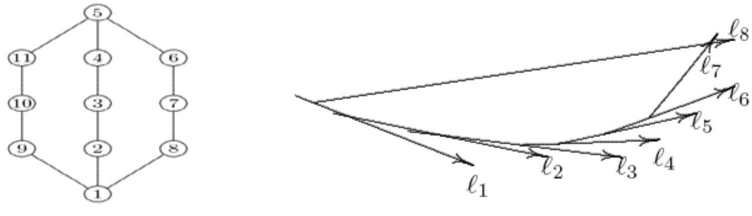


Figure 6. The graph G(5,5,5)

The next theorem gives us a criterion in determining whether a graph G does not belong to the class of wedge graphs.

Theorem 3.4

If a graph G contains an induced subgraph H which is not a wedge graph, then G itself is not a wedge graph.

Corollary 3.5

A graph G is not a wedge graph if it contains G(5,5,5) as an induced subgraph.

JOIN WEDGE OF GRAPHS

Definition 2.4.2

Let G and H be graphs with disjoint vertex-sets, their join or sum, denoted by G+H, is formed by taking G∪H and adding all edges of the form (x,y) where x is in G and y is in H.

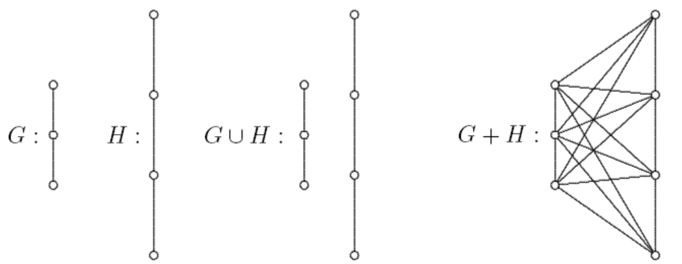


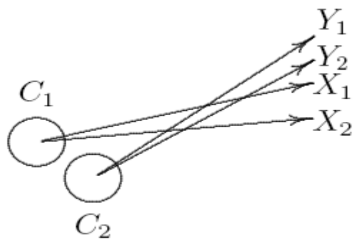
Figure 7. The union and join of two graphs

Theorem 4.2.1

If a finite number of graphs G_1, G_2,...,G_k is a wedge graph then their sum or join ($G_1 + G_2 + ... + G_k$) is also a wedge graph.

Proof. The proof is based on the Theorem by Gervacio [3] which states that the finite join of trees is a half-line intersection graph. The statement is true when k = 1. Let k > 1 and assume that the theorem is true for k-1. Given k number of graphs and an arbitrary θ with $0 < \theta < \pi$, let $\beta = \frac{1}{4}\theta$. By hypothesis of induction, the join $G_1, G_2,...,G_{k-1}$ is contained in a β-slice of the plane, S_β that is $G_1+G_2+...+G_{k-1} \cong \Omega(\mathscr{L}_1)$. Let C_1 be a circle centered at the vertex of S_β which encloses all the points of intersection of the half-lines in $\mathscr{L}_1$. Let $\mathscr{L}_2$ be the family of half-lines contained in a θ_k-slice of the plane, $S_{\theta k}$, that is $G_k \cong \Omega(\mathscr{L}_2)$ and C2 be a circle centered at the vertex of $S_{\theta k}$

enclosing all the points of intersection of the half-lines in $\mathscr{L}_2$. By suitably translating and rotating $S_{\theta k}$ together with the half-lines in it, we can form the join $G_1+G_2+...+G_k$. Let us assume that the boundaries of S_β are the half-lines X_1 and X_2. Let α_1 be the angle of inclination of X_1 and let $\alpha_1 + \beta$ be the angle of inclination of X2. We position $S_{\theta k}$ such that C1 and C2 do not intersect. Let Y_1 and Y_2 be the half-lines bounding $S_{\theta k}$. Let α_2 be the angle of inclination of Y_1 and let $\alpha_2 + \theta_k$ be the angle of inclination of Y_2. By rotating $S_{\theta k}$ about its vertex, we can make $\alpha_2 = \alpha_1 + \beta$. Then every half-line in $\mathscr{L}_1$ will intersect all the half-lines in $\mathscr{L}_2$. Furthermore, $\mathscr{L}_1 \cup \mathscr{L}_2$ is clearly contained in a $\frac{3}{4}\theta$-slice of the plane, and hence in a θ-slice of the plane. □



The following statements are direct consequences of the above theorem because the graphs are obtained as sum of two wedge graphs.

Corollary 4.2.2

The plane graphs with triangle such as the Fan (F_n). Generalized Fan ($F_{2,n}$), and the Wheel (W_n, $n \geq 3$, are wedge graphs and hence, segment intersection graphs.

Proof. The Fan (F_n) by definition is the sum of a Path P_n and a single vertex (K_1) which are wedge graphs. An example is the Fan F_5 and its half-line representation shown below.

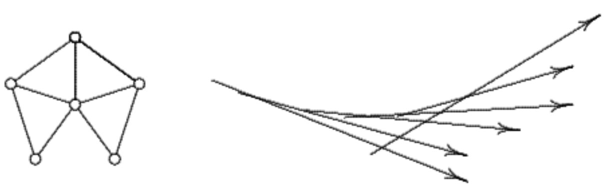


Figure 8. The Fan F_5 and its half-line representation

The Generalized Fan ($F_{2,n}$) is the sum of the Path P_n and K_2 which are wedge graphs.

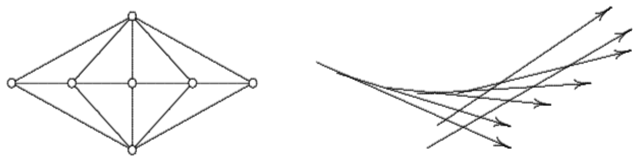


Figure 9. The Generalized Fan $F_{2,5}$ and its half-line representation

The Wheel (W_n, $n \geq 3$) is the sum of a cycle Cn and a single vertex which are wedge graphs. By Theorem 4.2.1, W_n, $n \geq 3$ is a wedge graph.

Corollary 4.2.3

Let m>2 and n>2. The non-planar graphs such as the Generalized Fan ($F_{m,n}$) and Complete Bipartite Graph ($K_{m,n}$) are wedge graphs and hence, segment intersection graphs.

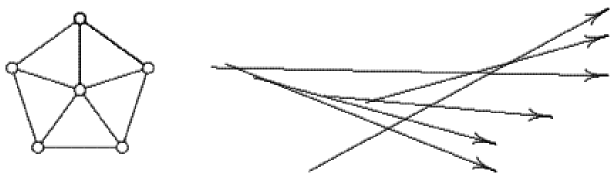


Figure 10. The Wheel W_5 and its half-line representation

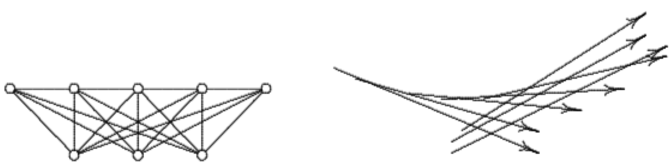


Figure 11. The Generalized Fan $F_{2,5}$ and its half-line representation

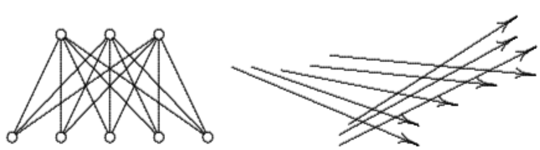


Figure 12. The Complete Bipartite Graph $K_{2,5}$ and its half-line representation

POWERS OF PATHS, FANS AND GENERALIZED FANS

Recall that the kth power of a graph G, denoted G^k is the graph with vertex set same as that of G and an edge (x,y) whenever $1 \leq d(x,y) \leq k$.

Consider the square of a path P_n. If the vertex set of P_n is $V(P_n)$ = {1,2,...,n} then $V(P_n^k)$ = $V(P_n)$ and the edge set is {[1,2],[2,3],...,[n-1,n], [2,4], [3,5],...,[n-2,n]}.

A pictorial representation of P_5^2 is shown in the figure below.

Figure 13. The square of the path P_5 and its half-line representation

The next theorem shows that the square of any path is a wedge graph.

Theorem 6.3.1

The square of the path P_n, denoted by P_n^2, is a wedge graph.

Remark 6.3.2

The (n-1)th power of P_n, is isomorphic to the complete graph of order ***n*** *which is a wedge graph.*

Theorem 6.3.3

The (n-2)th power of P_n, P_n^{n-2} for $n \geq 4$, is a wedge graph.

The result of Theorem 6.3.1 can be extended up to the kth power of P_n for any integer $k \leq n-1$.

Theorem 6.3.4

For any integer $k \leq n-1$, the kth power of Pn, P_n^k is a wedge graph.

The result of Theorem 6.3.4 is useful in establishing that the kth power of a Fan and Generalized Fan are wedge graphs.

Theorem 6.3.5

For any integer k≤n-1, the kth power of a Fan, F_n^k is a wedge graph.

Theorem 6.3.6

For any integer k≤n-1, the kth power of the generalized fan, $F_{m,n}^k$ is a wedge graph.

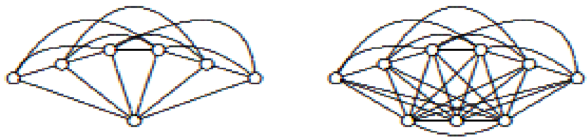


Figure 14. F_6^3 and $F_{3,6}^3$

POWER OF CYCLES, WHEELS AND STARS

Theorem 6.4.1

The square of a cycle, denoted by C_n^2 is a wedge graph.

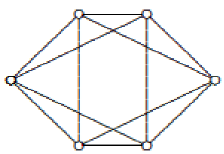


Figure 15. The square of C_6

Remark 6.4.2

For any integer k, such that $k \geq \frac{n-1}{2}$, $C_n^k \cong K_n$.

Theorem 6.4.3

For any integer $k \geq \frac{n-1}{2}$, the kth power of a cycle, C_n^k is a wedge graph.

We use Theorem 6.4.3 to establish that the kth power of a wheel and a star is a wedge graph.

Theorem 6.4.4

For any integer $k \geq \frac{n-1}{2}$, *the kth powerof a wheel,* W_n^k *is a wedge graph.*

Theorem 6.4.5

For any integer $k \geq \frac{n-1}{2}$, *the kth powerof a Star,* S_{m+1}^k *is a wedge graph.*

The question of whether the class of wedge graphs is closed under the operation of taking powers of graphs is not yet established and is still an open problem.

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